18.152 PROBLEM SET 6 SOLUTIONS

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1. Problem 3

In Lecture 8, we studied a variant of Problem 3 in dimension 2. The main difference between $\mathbb{R}^2$ and $\mathbb{R}^6$ arises from the Sobolev Embedding theorem. Let us first state the version we need for this problem

Theorem 1.1 (The Sobolev Inequality). *Let $\overline{\Omega} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Then for any smooth function $f \in C^\infty(\overline{\Omega})$ with $f|_{\partial\Omega} \equiv 0$, we have*

$$\|f\|_{L^q(\Omega)} \leqslant C(\Omega, p, n)\|\nabla f\|_{L^p(\Omega)} \text{ with } \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

for any $1 \leqslant p < n$. When $n < p \leqslant \infty$, we have

$$\|f\|_{L^\infty(\Omega)} \leqslant C(\Omega, p, n)\|\nabla f\|_{L^p(\Omega)}.$$

The Sobolev inequality allows us to trade derivatives of f to increase the index p. Suppose $p = 2$ and $n = 6$, then we can estimate L^q-norm of f with $q = 3 > 2$. To increase q further, we have to make use of higher derivative of f:

Theorem 1.2 (The Sobolev Inequality II). *Let $\overline{\Omega} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Then for any $m \geqslant 1$ and any smooth function $f \in C^\infty(\overline{\Omega})$ with $\nabla^k f|_{\partial\Omega} \equiv 0$, $0 \leqslant k \leqslant m-1$, we have*

$$\|f\|_{L^q(\Omega)} \leqslant C(\Omega, p, n, m)\|\nabla^m f\|_{L^p(\Omega)} \text{ with } \frac{1}{q} = \frac{1}{p} - \frac{m}{n}.$$

for any $1 \leqslant p < \frac{n}{m}$. When $p > \frac{n}{m}$, we have

$$\|f\|_{L^\infty(\Omega)} \leqslant C(\Omega, p, n, m)\|\nabla^m f\|_{L^p(\Omega)}.$$

Proof. By repeatedly applying Theorem 1.1, we obtain that

$$\|\nabla^k f\|_{L^{q_k}(\Omega)} \leqslant C(\Omega, q_{k+1}, n)\|\nabla^{k+1} f\|_{L^{q_{k+1}}(\Omega)} \text{ with } \frac{1}{q_k} = \frac{1}{p} - \frac{m-k}{n},$$

for any $0 \leqslant k \leqslant m-1$ if $1 \leqslant p < \frac{n}{m}$. Then we combine these estimates together.

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If $p > \frac{n}{m}$, at some intermediate step, we will be able to estimate

$$\|\nabla^k f\|_{L^\infty(\Omega)}$$

for some $0 \leqslant k \leqslant m-1$. Then we apply the second part of Theorem 1.1 to continue. □

As a result, in dimension 2,

$$\|f\|_{L^\infty(\Omega)} \leqslant \|\nabla^2 f\|_{L^2(\Omega)},$$

while in dimension 6,

$$\|f\|_{L^6(\Omega)} \leqslant \|\nabla^2 f\|_{L^2(\Omega)} \text{ and } \|f\|_{L^\infty(\Omega)} \leqslant \|\nabla^4 f\|_{L^2(\Omega)}.$$

Remark 1.3. *For the critical index $p = n$, the estimate*

$$\|f\|_{L^\infty(\Omega)} \leqslant \|\nabla f\|_{L^n(\Omega)},$$

does not hold.

Solution to Problem 3. We follow Lecture 8 and divide the proof into 4 steps. Fix a large constant $T > 0$. We focus on the region $[0, T] \times \mathbb{R}^6$.

Step 1. The solution u vanishes outside $[0, T] \times B_{R+T}(0)$. This follows from the energy estimate. For any $(t_0, x_0) \notin [0, T] \times B_{R+T}(0)$, consider the integral

$$E_{x_0}(t) = \frac{1}{2}\int_{|x-x_0|\leqslant T-t} |\nabla u|^2 + |u_t|^2 \geqslant 0.$$

Then $E(0) = 0$ and $E'(t) \leqslant 0$, $0 \leqslant t \leqslant T$. As a result, $E(t_0) = 0$ and $u(t_0, x_0) = 0$ if $t_0 < T$. The case when $t_0 = T$ follows from the continuity of u.

Step 2. Now consider the global energy

$$E(t) := \frac{1}{2}\int_{\{t\}\times B_{R+T}(0)} |\nabla u|^2 + |u_t|^2 \geqslant 0.$$

Since $E(t)$ is conservative in time,

$$\frac{1}{2}\int_{\{t\}\times B_{R+T}(0)} |\nabla u|^2 \leqslant E(t) = E(0) = \frac{1}{2}\int_{\mathbb{R}^6} |\nabla g|^2 + |h|^2,$$

for any $0 \leqslant t \leqslant T$.

Step 3. Repeat *Step* 1 and *Step* 2 for spatial derivatives of u. Let $\alpha = (\alpha_1, \cdots, \alpha_6)$, $\alpha_i \in \mathbb{Z}_{\geqslant 0}$ be a multi-index with $|\alpha| := \sum \alpha_i$ and define

$$D^\alpha u = \partial_1^{\alpha_1} \cdots \partial_6^{\alpha_6} u.$$

Then for any multi-index α, $D^\alpha u$ solves the wave equation with initial data:

$$\begin{aligned}(D^\alpha u)_{tt} &= \Delta(D^\alpha u), x \in \mathbb{R}^6, t \geqslant 0,\\ D^\alpha u(x,0) &= D^\alpha g(x),\\ (D^\alpha u(x,0))_t &= D^\alpha h(x).\end{aligned}$$

As a result, *Step* 2 implies that

$$\frac{1}{2}\int_{\{t\}\times B_{R+T}(0)} |\nabla D^\alpha u|^2 \leqslant \frac{1}{2}\int_{\mathbb{R}^6} |\nabla^{|\alpha|+1} g|^2 + |\nabla^{|\alpha|} h|^2.$$

Apply this estimate for any multi-index with $|\alpha| \leqslant 3$, we conclude that

$$\int_{\{t\}\times B_{R+T}(0)} |\nabla^4 u|^2 \leqslant C(\nabla^4 g, \nabla^3 h).$$

Step 4. Now we apply Theorem 1.2 with $\Omega = B_{R+T}(0)$, then

$$\|u\|_{L^\infty(\{t\}\times B_{R+T}(0))} \leqslant \|\nabla^4 u\|_{L^2(\{t\}\times B_{R+T}(0))} \leqslant C(T, R, \nabla^4 g, \nabla^3 h),$$

for any $0 \leqslant t \leqslant T$. □

2. Problem 4(b)

Many students figured out Problem 4(a), but most of you didn't realize how to apply 4(a) to 4(b). In fact, Problem 4(b) do not require any new computations.

Sketch of Problem 4(b). Problem 4(a) was stated for the initial data:

$$g(x) = \sum_{i=1}^{\infty} g_i w_i(x), \qquad h(x) = \sum_{i=1}^{\infty} h_i w_i(x).$$

Let us apply Problem 4(a) to the initial data ($k < l$):

$$g_{k,l}(x) := \sum_{i=k+1}^{l} g_i w_i(x), \qquad h_{k,l}(x) := \sum_{i=k+1}^{l} h_i w_i(x).$$

Then the function $u^k - u^l$ solves the wave equation with:

$$(u^k - u^l)(x,0) = g_{k,l}(x), \qquad (u^k - u^l)_t(x,0) = h_{k,l}(x).$$

Now Problem 4(a) implies that

$$\begin{aligned}\|\nabla(u^k - u^l)(\cdot,t)\|^2_{L^2(\Omega)} &\leqslant \|\nabla g_{k,l}\|^2_{L^2(\Omega)} + \|h_{k,l}\|^2_{L^2(\Omega)}\\ &= \sum_{i=k+1}^{l} |g_i|^2\lambda_i + \sum_{i=k+1}^{l} |h_i|^2.\end{aligned}$$

On the other hand,

$$\|\nabla g\|_2^2 = \sum_{i=1}^{\infty} |g_i|^2 \lambda_i, \qquad \|h\|_2^2 = \sum_{i=1}^{\infty} |h_i|^2,$$

so

$$\lim_{k,l\to\infty} \sum_{i=k+1}^{l} |g_i|^2 \lambda_i + \sum_{i=k+1}^{l} |h_i|^2 = 0. \qquad \square$$

3. Problem 4(c)

Solution to Problem 4(c). Given $\epsilon > 0$, we choose a smooth function $\chi : \Omega \to \mathbb{R}$ such that $\chi(x) = 0$ if $d(x, \partial\Omega) \geqslant 2\epsilon$ and $\chi(x) = 0$ if $d(x, \partial\Omega) \leqslant 0$. Then, given $v : \Omega \to \mathbb{R}$ we have

$$\|\nabla^k(\chi v)\|_{L^2} \leqslant C \sum_{i=0}^{k} \|\nabla^i v\|_{L^2},$$

for some $C = C(\chi, k)$. If χv is smooth, then $\chi v \in C_0^\infty$. Hence, the Sobolev inequality implies that there exists a fixed $m \in \mathbb{N}$ such that

$$\|v\|_{C^{k-m}(\Omega_{2\epsilon})} \leqslant \|\chi v\|_{C^{k-m}(\Omega)} \leqslant \|\nabla^k(\chi v)\|_{L^2} \leqslant C \sum_{i=0}^{k} \|\nabla^i v\|_{L^2},$$

where $\Omega_{2\epsilon} = \{x \in \Omega : d(x, \partial\Omega) \geqslant 2\epsilon\}$.

On the other hand, by using the idea of the problem 4 (b), we have

$$\lim_{\min\{k,l\}\to+\infty} \sum_{i=0}^{j} \|\nabla^i(u^k - u^l)\|_{L^2} = 0.$$

Therefore, for each $j \in \mathbb{N}$ we have

$$\lim_{\min\{k,l\}\to+\infty} \|u^k - u^l\|_{C^{j-m}(\Omega_{2\epsilon})} = 0,$$

namely the limit $u = \lim u^k$ is of class $C^{j-m}(\Omega_{2\epsilon})$ for each j and ϵ. Thus, u is smooth. In addition, $u_{tt} - \Delta u = \lim(u^k_{tt} - \Delta u^k) = 0$ holds in Ω, namely u is a solution to the wave equation. Moreover, the Cauchy conditions $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$ hold by definition. In addition, $u^k = 0$ on $\partial\Omega$ implies $u = 0$ on $\partial\Omega$. However, we need to show that $u \in C^0(\overline{\Omega})$, which we would not discuss in this course. $\square$